

ON NILPOTENT COMMUTING VARIETIES AND COHOMOLOGY OF FROBENIUS KERNELS

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ABSTRACT. The paper studies the dimensions of irreducible components of commuting varieties of (restricted) nilpotent r -tuples in a classical Lie algebra \mathfrak{g} defined over an algebraically closed field k . As applications, we obtain some new results on the structure of the (even) cohomology ring of Frobenius kernels G_r for each $r \geq 1$, where G is the simply connected, simple algebraic group such that $\mathrm{Lie}(G) = \mathfrak{g}$. Explicit calculations for rank two groups are also presented.

1. INTRODUCTION

1.1. Let \mathfrak{g} be a classical Lie algebra defined over an algebraically closed field k of characteristic $p > 0$. Denote $\mathcal{N}_1 = \{x \in \mathfrak{g} : x^{[p]} = 0\}$, the restricted nullcone of \mathfrak{g} . Note that \mathcal{N}_1 coincides with the nilpotent cone \mathcal{N} of \mathfrak{g} when $p \geq h$, the Coxeter number of \mathfrak{g} . In this paper, we study the dimension and related properties of the commuting variety

$$C_r(\mathcal{N}_1) = \{(x_1, \dots, x_r) \in \mathcal{N}_1^r : [x_i, x_j] = 0, 1 \leq i, j \leq r\}.$$

It is well known that for $r = 2$ and $p \geq h$, such commuting variety is completely described by Premet in [Pr]. Explicitly, he showed that $C_2(\mathcal{N})$ has pure dimension $\dim \mathfrak{g}$, each irreducible component is of the form $\overline{G \cdot (x, z(x) \cap \mathcal{N})}$ for some distinguished nilpotent element x , where $z(x)$ is the centralizer of x in \mathfrak{g} . In a part of the Ph.D. dissertation, the author proved that if \mathfrak{g} is either \mathfrak{sl}_2 or \mathfrak{sl}_3 , then $C_r(\mathcal{N})$ is irreducible for each $r \geq 1$ [Ngo2]. Recently, Šivic and the author studied the reducibility of $C_r(\mathcal{N})$ for type A and all $r \geq 1$ [NS]. Among other results, we proved that $C_r(\mathcal{N})$ is reducible for $r \geq 4$ and $\mathrm{rank}(\mathfrak{g}) \geq 3$. For $r = 3$, it was shown to be irreducible for $\mathrm{rank}(\mathfrak{g}) \leq 5$. Very little is known about $C_r(\mathcal{N}_1)$ for $p < h$. In fact, one could only find in literature related results for $r = 1$ [CLNP][UGA] or $r = 2, p = 2, \mathfrak{g} = \mathfrak{sl}_n$ [L].

Our motivation for studying such commuting varieties is to investigate cohomological properties of Frobenius kernels of G . The r -th Frobenius kernels G_r , for all $r \geq 1$, are infinitesimal subgroups of G whose coordinate algebras are finite dimensional and local. These groups play a fundamental role in relating cohomology theory of finite groups to that of reductive group schemes [Jan1]. However, cohomology theory for these objects are not well understood except few special cases. In the case $r = 1$, the first Frobenius kernel G_1 is a familiar object and received considerable interest from representation theorists due to the equivalence between the category of G_1 -modules and that of the restricted Lie algebra $(\mathfrak{g} = \mathrm{Lie}(G), [p])$ -modules, see for example [Jan1]. For higher values of r , the problem on computing the cohomology of G_r turns out to be complicated. Bendel, Nakano, and Pillen have made some progress in the two papers [BNP1][BNP2] where they explicitly calculated the first and second degrees of $H^i(G_r, H^0(\lambda))$ with $H^0(\lambda) = \mathrm{ind}_B^G(\lambda)$ the induced module of the highest weight λ . In the special case when $G = SL_2$, the author computed $H^i(G_r, H^0(\lambda))$ for each $i, r \geq 1$ and dominant weight λ [Ngo1]. In general, no conjecture has been made. Geometrically, Suslin, Friedlander, and Bendel, demonstrated that the spectrum of the cohomology ring $H^{2\bullet}(G_r, k)$ can be identified with the commuting variety $C_r(\mathcal{N}_1)$ [SFB1]. This groundbreaking result deduces the study of cohomology for Frobenius kernels to that of the variety $C_r(\mathcal{N}_1)$.

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1.2. The paper is structured as follows. We first prove that Premet's result does not hold for $C_r(\mathcal{N})$ in general. In other words, we point out that for all r larger than some constant depending on the type and rank of \mathfrak{g} , the commuting variety $C_r(\mathcal{N})$ is not equidimensional. Our main ingredient is the G -saturation variety $\mathfrak{V}_r = G \cdot \mathfrak{w}^r$ with \mathfrak{w} a fixed nilpotent commutative subalgebra of \mathfrak{g} defined at the beginning of Section 3.2. The dimension of \mathfrak{V}_r gives sharp lower bounds for those of $C_r(\mathfrak{u}_1)$ and $C_r(\mathcal{N}_1)$. It also allows us to compute, for \mathfrak{g} of type A and C , the dimension of the commuting variety over the square zero set $\mathfrak{D}_2 = \{x \in \mathfrak{g} : (i(x))^2 = 0\}$, where the inclusion $i : \mathfrak{g} \hookrightarrow \mathfrak{gl}_n$ is the natural representation of \mathfrak{g} (defined in 2.1). Consequently, we calculate the dimension of $C_r(\mathcal{N}_1)$ for \mathfrak{g} of rank 2.

The rest of the paper, as applications of the previous part, is devoted to explore the structure of the cohomology ring $H^\bullet(G_r, k)$ and complexity of G_r -module M . In particular, we first show that if $G = SL_2$, then the graded commutative ring $H^\bullet(G_r, k)$ is Cohen-Macaulay for each $r \geq 1$. This result significantly strengthens the one in [Ngo1] where the author proved that the commutative ring $H^{2\bullet}(G_r, k)_{\text{red}}$ is Cohen-Macaulay. However, this special property can not be generalized for arbitrary G , as we show in the later part that $H^\bullet(G_r, k)$ is, in general, not equidimensional. We are also able to provide a universal lower bound (depending only on r and $\text{rank}(\mathfrak{g})$) for the Krull dimension of $H^\bullet(B_r, k)$ and $H^\bullet(G_r, k)$ and then compute exactly this amount for G of rank 2. Note that our last results are much stronger than the computations of Kaneda et. al. on the Krull dimension of B_r cohomology ring for SL_3 [KSTY, 4.7]. Finally, we obtain some properties of the complexity, $c_{G_r}(M)$, of a module over G_r .

2. NOTATION

2.1. Representation theory. Let k be an algebraically closed field of characteristic $p > 0$. Let \mathfrak{g} be a classical Lie algebra over k , i.e. \mathfrak{g} is of type A, B, C , or D . Throughout the paper, we always assume p is a good prime for \mathfrak{g} , i.e. p is arbitrary for type A and it is greater than 2 for other types (see details for exceptional types in [Jan2, 2.6]), unless otherwise stated. To be convenient for later usage, we give an explicit description of these classical Lie algebras as subalgebras of the general linear algebra \mathfrak{gl}_n for some $n > 0$ as follows. For type A_ℓ , (i.e. $\mathfrak{sl}_{\ell+1}$) it is exactly the space of traceless $(\ell+1) \times (\ell+1)$ matrices. For other types, our Lie algebras are defined by the same strategy as in [Hum, 1.2] but using different nondegenerate (skew)-symmetric bilinear forms. Indeed, let J_ℓ be the anti-identity $\ell \times \ell$ matrix (consisting of 1's on the anti diagonal and 0's elsewhere). It is easy to see that the forms

$$\begin{pmatrix} 0 & J_\ell \\ -J_\ell & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & J_\ell \\ J_\ell & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & J_\ell \\ 0 & 1 & 0 \\ J_\ell & 0 & 0 \end{pmatrix}$$

are symplectic and orthogonal bilinear forms of $\mathfrak{sp}_{2\ell}$, $\mathfrak{so}_{2\ell}$ and $\mathfrak{so}_{2\ell+1}$. Now for each matrix m in \mathfrak{gl}_ℓ , denote m^J the matrix reflected over the anti-diagonal. Similarly as in [Hum, 1.2], m^t is the transposed matrix of m . Finally classical Lie algebras (other than type A) can be defined as follows:

- Type C_ℓ or D_ℓ (i.e. $\mathfrak{sp}_{2\ell}$ or $\mathfrak{so}_{2\ell}$): is the space of matrices of the form

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

satisfying $m_{ij} \in \mathfrak{gl}_\ell$, m_{12}, m_{21} are (skew) symmetric over the anti-diagonal, and $m_{11} = \pm m_{22}^J$.

- Type B_ℓ (i.e. $\mathfrak{so}_{2\ell+1}$) is the space of matrices of the form

$$\begin{pmatrix} m_{11} & b_1 & m_{12} \\ c_1 & 0 & c_2 \\ m_{21} & b_2 & m_{22} \end{pmatrix}$$

where $m_{ij} \in \mathfrak{gl}_\ell$, b_1, b_2 and c_1, c_2 are column and row vectors in k^ℓ such that m_{12}, m_{21} are skew symmetric over the anti-diagonal, $m_{11} = -m_{22}^J$, $J_\ell b_1 = -c_2^t$, and $J_\ell b_2 = -c_1^t$.

The above construction of classical linear Lie algebras implies an inclusion $i : \mathfrak{g} \hookrightarrow \mathfrak{gl}_n$ for some $n > 0$. Fix a Borel subalgebra \mathfrak{b} consisting of upper triangular matrices in \mathfrak{g} , and a Cartan subalgebra \mathfrak{t} consisting of diagonal matrices in \mathfrak{g} . Now let G be a simply connected, simple algebraic group over k , stabilizing the aforementioned bilinear forms, such that $\text{Lie}(G) = \mathfrak{g}$, (explicit definition of G could be found in [MT, §1.2]). Fix a maximal torus T of G , let $B \subset G$ be the Borel subgroup of G containing T satisfying $\text{Lie}(B) = \mathfrak{b}$ and $\text{Lie}(T) = \mathfrak{t}$. Let $U \subset B$ be the unipotent radical of B , then $\text{Lie}(U) = \mathfrak{u}$, consisting of strictly upper triangular matrices. From now on, the symbol \otimes means the tensor product over the field k , unless otherwise stated. Suppose H is an algebraic group over k and M is a (rational) module of H . Denote by M^H the submodule consisting of all the fixed points of M under the H -action.

For each positive integer r , let $F_r : G \rightarrow G$ be the r -th Frobenius morphism, see for example [Jan1, I.9]. The scheme-theoretic kernel $G_r = \ker(F_r)$ is called the r -th Frobenius kernel of G . Given a closed subgroup (scheme) H of G , write H_r for the scheme-theoretic kernel of the restriction $F_r : H \rightarrow H$. In other words, we have

$$H_r = H \cap G_r.$$

Given a rational G -module M , write $M^{(r)}$ for the module obtained by twisting the structure map for M by F_r . Note that G_r acts trivially on $M^{(r)}$. Conversely, if N is a G -module on which G_r acts trivially, then there is a unique G -module M with $N = M^{(r)}$. We denote the module M by $N^{(-r)}$. Let M be a B -module. Then the induced G -module can be defined as

$$\text{ind}_B^G M = (k[G] \otimes M)^B.$$

2.2. Geometry. Let R be a commutative Noetherian ring with identity. We use R_{red} to denote the reduced ring $R/\text{Nilrad } R$ where $\text{Nilrad } R$ is the radical ideal of 0 in R , which consists of all nilpotent elements of R . Let $\text{MaxSpec}(R)$ be the spectrum of all maximal ideals of R . This set is a topological space under the Zariski topology. The notation $\dim(-)$ will be interchangeably used as the dimension of a variety or the Krull dimension of a ring.

Let \mathcal{N} be the nilpotent cone of \mathfrak{g} , consisting of all nilpotent elements in \mathfrak{g} . The adjoint action of G on \mathfrak{g} stabilizes \mathcal{N} and is denoted by “ \cdot ”. For each $x \in \mathcal{N}$, denote $\mathcal{O}_x = G \cdot x$ the orbit of x under the dot action of G . Let x_{reg} be a fixed regular nilpotent element and z_{reg} be its centralizer in \mathfrak{g} . It is well known that $z_{\text{reg}} \subset \mathcal{N}$, $\dim z_{\text{reg}} = \text{rank } \mathfrak{g} =: \ell$, and the regular orbit $\mathcal{O}_{\text{reg}} = G \cdot x_{\text{reg}}$ is dense in \mathcal{N} . The restricted nullcone \mathcal{N}_1 of \mathfrak{g} is defined as a subvariety of \mathcal{N} satisfying

$$x \in \mathcal{N}_1 \Leftrightarrow x^{[p]} = 0$$

where $(-)^{[p]}$ is the p -power operation of the restricted Lie algebra \mathfrak{g} . Since our classical Lie algebras could be embedded into \mathfrak{gl}_n for some $n > 0$, one may identify $x^{[p]} = i(x)^p$ for $x \in \mathfrak{g}$. Hence, for $p \geq h$, the Coxeter number of \mathfrak{g} , we have $\mathcal{N}_1 = \mathcal{N}$, see for instance [CLNP, §1]. Complete description of \mathcal{N}_1 is referred to the paper of Carlson, Lin, Nakano, and Parshall [CLNP]. Set $\mathfrak{u}_1 = \mathcal{N}_1 \cap \mathfrak{u}$. It follows that $\mathfrak{u}_1 = \mathfrak{u}$ whenever $p \geq h$.

3. COMMUTING VARIETIES

Suppose V is a closed affine subvariety of \mathfrak{g} . We define the commuting variety of r -tuples over V as follows

$$C_r(V) = \{(x_1, \dots, x_r) \in V^r \mid [x_i, x_j] = 0, 1 \leq i \leq j \leq r\}.$$

We will just call it the commuting variety over V for short. In case when $V = \mathcal{N}$ (or \mathcal{N}_1), we call $C_r(V)$ the (restricted) nilpotent commuting variety of \mathfrak{g} . For more details of such varieties, one can refer to [Ngo2].

3.1. Irreducible component associated to regular nilpotent elements. One has seen such an irreducible component in the case when $r = 2$, see for example [Pr]. We show here that its generalized version for arbitrary r is still an irreducible of $C_r(\mathcal{N})$. Similarly, we point out an irreducible component for $C_r(\mathfrak{u})$. The dimension of these components gives some criterion for irreducibility and equidimensionality.

Proposition 3.1.1. *For each $r \geq 1$, the closed subvariety $V_{\text{reg}} := \overline{G \cdot (x_{\text{reg}}, z_{\text{reg}}, \dots, z_{\text{reg}})}$ is an irreducible component of $C_r(\mathcal{N})$ whose dimension is $\dim \mathcal{N} + (r - 1)\ell$.*

Proof. First note that V_{reg} is irreducible as it is the image of the surjective morphism

$$\begin{aligned} m : G \times z_{\text{reg}}^{r-1} &\rightarrow V_{\text{reg}} \\ (g, x_1, \dots, x_{r-1}) &\mapsto g \cdot (x_{\text{reg}}, x_1, \dots, x_{r-1}) \end{aligned}$$

for all $g \in G$ and $x_i \in z_{\text{reg}}$. Now consider the projection from $C_r(\mathcal{N})$ to its first component

$$\begin{aligned} \rho : C_r(\mathcal{N}) &\rightarrow \mathcal{N}, \\ (x_1, \dots, x_r) &\mapsto x_1. \end{aligned}$$

Since the orbit \mathcal{O}_{reg} is open in \mathcal{N} , so is its preimage $\rho^{-1}(\mathcal{O}_{\text{reg}}) = G \cdot (x_{\text{reg}}, z_{\text{reg}}, \dots, z_{\text{reg}})$ (here we use the fact that z_{reg} is commutative). So the closure $\overline{\rho^{-1}(\mathcal{O}_{\text{reg}})} = V_{\text{reg}}$ is an irreducible component of $C_r(\mathcal{N})$.

Applying the theorem on the dimension of fibers to the restriction of $\rho : V_{\text{reg}} \rightarrow \mathcal{N}$, we have

$$\dim V_{\text{reg}} = \dim \mathcal{N} + \dim \rho^{-1}(x_{\text{reg}}) = \dim \mathcal{N} + (r - 1)\ell,$$

which completes our proof. \square

From now on, we call V_{reg} the irreducible component of $C_r(\mathcal{N})$ associated to regular nilpotent elements. Replacing G by B in the above argument, one obtains a similar result for $C_r(\mathfrak{u})$ as follows.

Proposition 3.1.2. *For each $r \geq 1$, the closed subvariety $\overline{B \cdot (x_{\text{reg}}, z_{\text{reg}}, \dots, z_{\text{reg}})}$ is an irreducible component of $C_r(\mathfrak{u})$ whose dimension is $\dim \mathfrak{u} + (r - 1)\ell$.*

An easy corollary immediately follows.

Corollary 3.1.3. *For each $r \geq 1$, if the commuting variety $C_r(\mathcal{N})$ (or $C_r(\mathfrak{u})$) is irreducible or equidimensional then its dimension is $\dim \mathcal{N} + (r - 1)\ell$ (or $\dim \mathfrak{u} + (r - 1)\ell$).*

3.2. Dimension of commuting varieties. We introduce here a special square zero vector space which plays the main role in our investigations on the dimensions of commuting varieties. Our construction below is based on the definition of classical linear Lie algebras in the beginning of Section 2. Explicitly, the space \mathfrak{w} is defined by matrices of the form

$$(1) \quad \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$$

where m satisfies the following

- Type A_n : If $n = 2\ell$ then m is an $\ell \times (\ell + 1)$ matrix, otherwise, if $n = 2\ell - 1$, then $m \in \mathfrak{gl}_\ell$,
- Type C_ℓ : $m \in \mathfrak{gl}_\ell$ and $m = m^J$,
- Type B_ℓ or D_ℓ : $m \in \mathfrak{gl}_\ell$ and $m = -m^J$.

It is not hard to see that \mathfrak{w} is a nilpotent Lie subalgebra of \mathfrak{g} (as it is square zero). Moreover, it is commutative and

$$(2) \quad \dim \mathfrak{w} = \begin{cases} \ell^2 & \text{if } \mathfrak{g} = \mathfrak{sl}_{2\ell}, \\ \ell(\ell + 1) & \text{if } \mathfrak{g} = \mathfrak{sl}_{2\ell+1}, \\ \frac{\ell^2 + \ell}{2} & \text{if } \mathfrak{g} = \mathfrak{sp}_{2\ell}, \\ \frac{\ell^2 - \ell}{2} & \text{else.} \end{cases}$$

Remark 3.2.1. One could realize \mathfrak{w} as the Lie algebra of a unipotent radical for some parabolic subgroup $P_{\mathfrak{w}}$ of G . Explicitly, suppose V is the natural representation of \mathfrak{g} with the basis $\{v_1, \dots, v_n\}$ where $n = \ell + 1$, $(2\ell + 1)$, or 2ℓ if \mathfrak{g} is of type A_ℓ (B_ℓ , or C_ℓ, D_ℓ). Let W be the subspace of V generated by the first $\lfloor \frac{n}{2} \rfloor$ basis vectors v'_i 's. Under the bilinear forms defined in Section 2, we have $0 \subset W \subset V$ is a totally isotropic flag, so Proposition 12.13 in [MT] states that the stabilizer of this flag in G is a parabolic subgroup, denoted by $P_{\mathfrak{w}}$. Simple calculations would show that $\mathfrak{w} = \text{Lie}(U)$ where U is the unipotent radical of $P_{\mathfrak{w}}$. (The reader should refer to Examples 12.4 and 17.9 in [MT] for the details).

Proposition 3.2.2. *The commuting variety $C_r(\mathfrak{u})$ is not equidimensional for*

$$r > \begin{cases} 2 + \frac{1}{\ell-1} & \text{if } \mathfrak{g} = \mathfrak{sl}_{2\ell} \text{ or } \mathfrak{sl}_{2\ell+1}, \\ 2 & \text{if } \mathfrak{g} = \mathfrak{sp}_{2\ell}, \\ 2 + \frac{2}{\ell-3} & \text{if } \mathfrak{g} = \mathfrak{so}_{2\ell}, \\ 2 + \frac{4}{\ell-3} & \text{if } \mathfrak{g} = \mathfrak{so}_{2\ell+1}. \end{cases}$$

Proof. As \mathfrak{w} is commutative, \mathfrak{w}^r is a subvariety of $C_r(\mathfrak{u})$. Easy computation shows that $\dim \mathfrak{w}^r (= r \dim \mathfrak{w})$ is greater than $\dim \mathfrak{u} + (r - 1)\ell$ for all r satisfying the hypothesis. Hence the result follows by Corollary 3.1.3. \square

Since $\mathfrak{w} \subset \mathfrak{u}_1$ for all $p \geq 2$, we further have $\mathfrak{w}^r \subset C_r(\mathfrak{u}_1)$. Thus, we obtain the following

Corollary 3.2.3. *For each $r \geq 1$ and for all prime $p \geq 2$ (not necessarily good prime), we always have*

$$\dim C_r(\mathfrak{u}_1) \geq r \dim \mathfrak{w}.$$

Before getting similar results for $C_r(\mathcal{N})$, we set $\mathfrak{V}_r = G \cdot \mathfrak{w}^r$ for each $r \geq 1$. It's easy to see that \mathfrak{V}_r is a closed subvariety of $C_r(\mathcal{N})$, see for example [Jan2, §8.7]. We now compute the dimension of \mathfrak{V}_r .

Proposition 3.2.4. *For each $r \geq 1$, one has*

$$\dim \mathfrak{V}_r = (r + 1) \dim \mathfrak{w} = \begin{cases} (r + 1)\ell^2 & \text{if } \mathfrak{g} = \mathfrak{sl}_{2\ell}, \\ (r + 1)\ell(\ell + 1) & \text{if } \mathfrak{g} = \mathfrak{sl}_{2\ell+1}, \\ (r + 1)\frac{\ell^2 + \ell}{2} & \text{if } \mathfrak{g} = \mathfrak{sp}_{2\ell}, \\ (r + 1)\frac{\ell^2 - \ell}{2} & \text{else.} \end{cases}$$

Proof. By (2), it suffices to prove the first equality. From the Remark 3.2.1, we have the moment map $\mathfrak{m} : G \times^{P_{\mathfrak{w}}} \mathfrak{w}^r \rightarrow \mathfrak{V}_r$. It follows that

$$\dim \mathfrak{V}_r \leq \dim (G/P_{\mathfrak{w}}) + r \dim \mathfrak{w} = (r + 1) \dim \mathfrak{w}.$$

On the other hand, let $x_{\mathfrak{w}}$ be an element in \mathfrak{w} which also belongs to a maximal orbit in \mathcal{N} intersecting with \mathfrak{w} . (The best candidate for such $x_{\mathfrak{w}}$ is the matrix form (1) where m is a diagonal matrix consisting of 1's or -1 's.) In particular, the corresponding partition of $x_{\mathfrak{w}}$ is $[2^s, 1^t]$ where s and t satisfy the following condition

- Type A_ℓ : $2s + t = \ell + 1$ with $t = 0$, or 1.
- Type C_ℓ : $s = \ell$ and $t = 0$.
- Type B_ℓ : $2s + t = 2\ell + 1$ with s even and $t = 1$, or 3.
- Type D_ℓ : $2s + t = 2\ell$ with s even and $t = 0$, or 2.

Using the Corollary 6.1.4 in [CM], one easily verifies that $\dim \mathcal{O}_{x_{\mathfrak{w}}} \geq 2 \dim \mathfrak{w}$. So

$$\dim \mathfrak{V}_r \geq \dim (G \cdot (x_{\mathfrak{w}}, \mathfrak{w}^{r-1})) = \dim (G \cdot x_{\mathfrak{w}}) + (r - 1) \dim \mathfrak{w} \geq (r + 1) \dim \mathfrak{w}.$$

Finally, we have obtained the equality. \square

Using the same argument as for $C_r(\mathfrak{u})$ and $C_r(\mathfrak{u}_1)$, one easily obtains the below properties.

Corollary 3.2.5. *The nilpotent commuting variety $C_r(\mathcal{N})$ is not equidimensional for*

$$r > \begin{cases} 3 & \text{if } \mathfrak{g} = \mathfrak{sl}_{2\ell+1}, \\ 3 + \frac{2}{\ell-1} & \text{if } \mathfrak{g} = \mathfrak{sl}_{2\ell}, \\ 3 + \frac{4}{\ell-1} & \text{if } \mathfrak{g} = \mathfrak{sp}_{2\ell}, \\ 3 + \frac{4}{\ell-3} & \text{if } \mathfrak{g} = \mathfrak{so}_{2\ell}, \\ 3 + \frac{8}{\ell-3} & \text{if } \mathfrak{g} = \mathfrak{so}_{2\ell+1}. \end{cases}$$

Proof. It is not hard to see that for these values of r , $\dim \mathfrak{V}_r > \dim V_{\text{reg}}$, so the result follows by Corollary 3.1.3. \square

Remark 3.2.6. The above result shows that Premet's result on equidimensionality of $C_2(\mathcal{N})$ [Pr] is not valid for $r \geq 3$. Our result also implies that the structure of $C_r(\mathcal{N})$ could be very complicated when r is large. Hence, the task of describing irreducible components of these varieties becomes challenging.

We next obtain a lower bound for the dimension of $C_r(\mathcal{N}_1)$. This bound depends on r, ℓ , and the type of \mathfrak{g} , not depend on p . Recall that we have been assuming that p is a good prime for G .

Corollary 3.2.7. *For each $r \geq 1$, one has*

$$\dim C_r(\mathcal{N}_1) \geq \dim \mathfrak{V}_r = \begin{cases} (r+1)\ell^2 & \text{if } \mathfrak{g} = \mathfrak{sl}_{2\ell}, \\ (r+1)\ell(\ell+1) & \text{if } \mathfrak{g} = \mathfrak{sl}_{2\ell+1}, \\ (r+1)\frac{\ell^2+\ell}{2} & \text{if } \mathfrak{g} = \mathfrak{sp}_{2\ell}, \\ (r+1)\frac{\ell^2-\ell}{2} & \text{else.} \end{cases}$$

Proof. The fact that \mathfrak{w} is square zero implies that it is always contained in \mathcal{N}_1 , and so $\mathfrak{V}_r \subset C_r(\mathcal{N}_1)$. Therefore, the inequality follows. \square

3.3. Recall from the Section 2.1 that we have the embedding $i : \mathfrak{g} \hookrightarrow \mathfrak{gl}_n$ for some $n > 0$. Now let $\mathfrak{D}_2 = \{x \in \mathfrak{g} : (i(x))^2 = 0\}$. We next compute the dimensions of $C_r(\mathfrak{D}_2 \cap \mathfrak{u})$ and $C_r(\mathfrak{D}_2)$. This deduces the dimensions of $C_r(\mathfrak{u}_1)$ and $C_r(\mathcal{N}_1)$ when \mathfrak{g} is of type A and $p = 2$. We assume for the rest of this section that \mathfrak{g} is of type A or C .

Suppose \mathcal{O}_2 is the maximal orbit in \mathfrak{D}_2 , i.e. $\mathfrak{D}_2 = \overline{\mathcal{O}_2}$. (Note that such orbit is not unique if \mathfrak{g} is of type D). In fact, $\mathcal{O}_2 = \mathcal{O}_{x_{\mathfrak{w}}}$ defined in the proof of Proposition 3.2.4. Hence, we recall that the partition of \mathcal{O}_2 is of the form $[2^s, 1^t]$ for some non-negative integers s, t . In particular, their values are following

- $\mathfrak{g} = \mathfrak{sl}_n$: $t = 1$ if n is odd, otherwise $t = 0$; hence $s = \frac{n-t}{2}$,
- $\mathfrak{g} = \mathfrak{sp}_{2\ell}$: $t = 0$ and $s = \ell$.

We review the dimension of \mathcal{O}_2 .

Lemma 3.3.1. *We have*

$$\dim \mathfrak{D}_2 = \dim \mathcal{O}_2 = \begin{cases} \frac{n^2-t^2}{2} & \text{if } \mathfrak{g} = \mathfrak{sl}_n, \\ \ell^2 + \ell & \text{if } \mathfrak{g} = \mathfrak{sp}_{2\ell}. \end{cases}$$

Proof. It easily follows from [CM, Corollary 6.1.4]. \square

We first compute the dimension of $C_r(\mathfrak{D}_2 \cap \mathfrak{u})$.

Lemma 3.3.2. *For each $r \geq 1$, we have*

$$\dim C_r(\mathfrak{D}_2 \cap \mathfrak{u}) = r \dim \mathfrak{w}$$

and \mathfrak{w}^r is an irreducible of maximal dimension in $C_r(\mathfrak{D}_2 \cap \mathfrak{u})$.

Proof. By [Jan2, Theorem 10.11], one has

$$\dim(\mathfrak{D}_2 \cap \mathfrak{u}) = \max_{\mathcal{O} \subset \mathfrak{D}_2} \{\dim(\mathcal{O} \cap \mathfrak{u})\} = \max_{\mathcal{O} \subset \mathfrak{D}_2} \left\{ \frac{1}{2} \dim \mathcal{O} \right\} = \frac{1}{2} \dim \mathfrak{D}_2.$$

It follows that $\dim C_r(\mathfrak{D}_2 \cap \mathfrak{u}) \leq \frac{r}{2} \dim \mathfrak{D}_2$. On the other hand, note that $\dim \mathfrak{D}_2 = 2 \dim \mathfrak{w}$ (from (2)). Hence the fact that \mathfrak{w}^r is a subset of $C_r(\mathfrak{D}_2 \cap \mathfrak{u})$ implies the lemma. \square

Now we prove the main result of this subsection.

Theorem 3.3.3. *For each $r \geq 1$, one has*

$$\dim C_r(\mathfrak{D}_2) = (r+1) \dim \mathfrak{w} = \begin{cases} (r+1) \lfloor \frac{n^2}{4} \rfloor & \text{if } \mathfrak{g} = \mathfrak{sl}_n, \\ (r+1) \frac{\ell^2 + \ell}{2} & \text{if } \mathfrak{g} = \mathfrak{sp}_{2\ell}. \end{cases}$$

Consequently, \mathfrak{V}_r is an irreducible component of maximal dimension in $C_r(\mathfrak{D}_2)$.

Proof. It is noted that $C_r(\mathfrak{D}_2) = G \cdot C_r(\mathfrak{D}_2 \cap \mathfrak{u})$ for each $r \geq 1$. It follows that

$$\begin{aligned} \dim C_r(\mathfrak{D}_2) &= \dim G - \dim N_G(C_r(\mathfrak{D}_2 \cap \mathfrak{u})) + \dim C_r(\mathfrak{D}_2 \cap \mathfrak{u}) \\ &= \dim G - \dim N_G((\mathfrak{D}_2 \cap \mathfrak{u})^r) + r \dim \mathfrak{w} \\ &= \dim G - \dim N_G(\mathfrak{D}_2 \cap \mathfrak{u}) + r \dim \mathfrak{w} \\ &= \dim[G \cdot (\mathfrak{D}_2 \cap \mathfrak{u})] - \dim(\mathfrak{D}_2 \cap \mathfrak{u}) + r \dim \mathfrak{w} \\ &= \dim \mathfrak{D}_2 - \dim \mathfrak{w} + r \dim \mathfrak{w} \\ &= (r+1) \dim \mathfrak{w} \end{aligned}$$

where $N_G(S) = \{g \in G : g \cdot S \subseteq S\}$, the normalizer of $S \subseteq \mathfrak{g}$. Thus, we have proved that $\dim C_r(\mathfrak{D}_2) = \dim \mathfrak{V}_r$ for each $r \geq 1$, hence the theorem follows from Proposition 3.2.4. \square

As an application, we are now able to compute the dimension of $C_r(\mathcal{N}_1)$ when $p = 2$ and \mathfrak{g} is of type A . Indeed, since $\mathcal{N}_1 = \mathfrak{D}_2$ when $p = 2$, we immediately have

Proposition 3.3.4. *Suppose $\mathfrak{g} = \mathfrak{sl}_n$ and $p = 2$. Then for each $r \geq 1$,*

$$\dim C_r(\mathcal{N}_1) = \dim C_r(\mathfrak{D}_2) = (r+1) \lfloor \frac{n^2}{4} \rfloor.$$

Remark 3.3.5. The last result not only points out the case when the equality in Corollary 3.2.7 occurs but also generalize a result in [L] on the dimension of $C_2(\mathcal{N}_1)$.

3.4. Rank 2 cases. Apply Theorem 3.3.3, we explicitly calculate the dimension of $C_r(\mathcal{N}_1)$ for \mathfrak{g} of rank 2, i.e. \mathfrak{g} is of type A_2 or C_2 . We keep assuming that the characteristic of k is a good prime. We first present the result for type A_2 .

Corollary 3.4.1. *Suppose \mathfrak{g} is of type A_2 and p is any prime. For each $r \geq 1$, one has*

$$\dim C(\mathfrak{u}_1) = \begin{cases} 2r+1 & \text{if } p \neq 2, \\ 2r & \text{if } p = 2, \end{cases}$$

and

$$\dim C_r(\mathcal{N}_1) = \begin{cases} 2r+4 & \text{if } p \neq 2, \\ 2r+2 & \text{if } p = 2. \end{cases}$$

Proof. Suppose $p > 2$. Then from Proposition 7.1.1 and Theorem 7.1.2 in [Ngo2], we have for each $r \geq 1$

$$\dim C_r(\mathfrak{u}_1) = \dim C_r(\mathfrak{u}) = 2r+1, \quad \dim C_r(\mathcal{N}_1) = \dim C_r(\mathcal{N}) = 2r+4.$$

Now if $p = 2$, then Lemma 3.3.2 and Proposition 3.3.4 give the desired results. \square

Corollary 3.4.2. *Suppose \mathfrak{g} is of type C_2 and $p \geq 3$. For $r \geq 1$, one has*

$$\dim C_r(\mathfrak{u}_1) = \begin{cases} 2r + 2 & \text{if } r = 1, p \neq 3, \\ 3r & \text{else,} \end{cases}$$

and

$$\dim C_r(\mathcal{N}_1) = \begin{cases} 2r + 6 & \text{if } r \leq 2, p \neq 3, \\ 3r + 3 & \text{else.} \end{cases}$$

Proof. First we consider the case when $p = 3$. By [UGA, Theorem 5.1], $\mathcal{N}_1 = \mathfrak{D}_2$ so that

$$\dim C_r(\mathfrak{u}_1) = 3r \quad , \quad \dim C_r(\mathcal{N}_1) = 3(r + 1)$$

by Lemma 3.3.2 and Theorem 3.3.3.

Now we assume $p > 3$. Then $\mathcal{N}_1 = \mathcal{N}$ and $\mathfrak{u}_1 = \mathfrak{u}$. Recall that \mathfrak{u} is the space of all matrices of the form

$$\begin{pmatrix} 0 & x & y & z \\ 0 & 0 & t & y \\ 0 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For $r \geq 2$, the commutator on these matrices implies that the variety $C_r(\mathfrak{u})$ is defined by polynomials

$$x_i t_j - x_j t_i, \quad x_i y_j - x_j y_i$$

for $1 \leq i < j \leq r$. It is then easy to see that $C_r(\mathfrak{u})$ is a closed subvariety of the one defined by polynomials $\{x_i t_j - x_j t_i\}_{1 \leq i, j \leq r}$ in the affine space \mathfrak{u}^r , which is the product of an affine space of dimension $2r$ (corresponding to free parameters y_i, z_j) and the determinantal variety of all 2×2 minors over the matrix

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_r \\ t_1 & t_2 & \cdots & t_r \end{pmatrix}.$$

Denote this product by \mathcal{P} , we then have by [Ngo2, Proposition 3.2.2], $\dim(\mathcal{P}) = 3r + 1$. Now as \mathcal{P} is irreducible and $C_r(\mathfrak{u})$ is a proper subvariety of \mathcal{P} , we obtain $\dim C_r(\mathfrak{u}) < 3r + 1$. On the other hand, $\dim C_r(\mathfrak{u}) \geq \dim C_r(\mathfrak{D}_2 \cap \mathfrak{u}) = 3r$. Thus, we have shown that $\dim C_r(\mathfrak{u}) = 3r$.

Next, as \mathcal{N} contains finitely many orbits we decompose

$$C_r(\mathcal{N}) = V_{\text{reg}} \cup \overline{G \cdot (x_{[2,2]}, C_{r-1}(z(x_{[2,2]}) \cap \mathcal{N}))} \cup \overline{G \cdot (x_{[2,1,1]}, C_{r-1}(z(x_{[2,1,1]}) \cap \mathcal{N}))}$$

where $x_{[2,2]}$ (or $x_{[2,1,1]}$) is a representative in the orbit of partition $[2, 2]$ (or $[2, 1, 1]$). Hence, any irreducible component of $C_r(\mathcal{N})$, other than V_{reg} , lies in one of the last two subvarieties. On the other hand, by similar argument as in [Pr, Proposition 2.1], we have $GL_r(k)$ acting on $C_r(\mathcal{N})$ as follows

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} \end{pmatrix} \bullet (x_1, \dots, x_r) = \left(\sum_{i=1}^r a_{1i} x_i, \dots, \sum_{i=1}^r a_{ri} x_i \right)$$

for all $(x_1, \dots, x_r) \in C_r(\mathcal{N})$. In particular, suppose V is an irreducible component of $C_r(\mathcal{N})$. Then any permutation of $(x_1, \dots, x_r) \in V$ is also in V . This indicates that if $V \subseteq \overline{G \cdot (x, C_{r-1}(z(x) \cap \mathcal{N}))}$ then V must be in $C_r(\overline{\mathcal{O}_x})$. Hence, the dimension of $C_r(\mathcal{N})$ is in fact the maximum of the set $\{\dim V_{\text{reg}}, \dim C_r(\overline{\mathcal{O}_{[2,2]}}), \dim C_r(\overline{\mathcal{O}_{[2,1,1]}})\}$. We already know that the first amount is $2r + 6$ (by Proposition 3.1.1). Since $C_r(\overline{\mathcal{O}_{[2,1,1]}}) \subset C_r(\overline{\mathcal{O}_{[2,2]}})$, and $\dim C_r(\overline{\mathcal{O}_{[2,2]}}) = \dim C_r(\mathfrak{D}_2) = 3(r + 1)$, we finally obtain

$$\dim C_r(\mathcal{N}) = \max\{2r + 6, 3r + 3\}.$$

This completes our proof. \square

4. STRUCTURE OF THE COHOMOLOGY RING OF FROBENIUS KERNELS

We keep assuming that p is a good prime for G . For each $r \geq 1$, let

$$H^\bullet(G_r, k) = \bigoplus_{i \geq 0} H^i(G_r, k) \quad , \quad H^{2\bullet}(G_r, k) = \bigoplus_{i \geq 0} H^{2i}(G_r, k)$$

where the latter is usually called the even cohomology ring of G_r . It is well known that $H^\bullet(G_r, k)$ is a graded commutative k -algebra. This section is aimed to answer the question on whether or not this G_r -cohomology ring is Cohen-Macaulay. This question is motivated from the conjectures in [Ngo1, §7.2]. Explicitly, we show that $H^\bullet((SL_2)_r, k)$ is Cohen-Macaulay for all $r \geq 1$. Some relation about Cohen-Macaulayness of U_r - and B_r -cohomology rings is also obtained. Finally, we apply some properties of commuting varieties in the previous section to point out the values of r for which the ring $H^\bullet(G_r, k)$ is not Cohen-Macaulay.

We begin by recalling some distinguished features of a Cohen-Macaulay graded commutative ring.

Proposition 4.0.3. [Ben, Proposition 2.5.1] *Let $R = \bigoplus_{i \geq 0} R_i$ be a finitely generated graded commutative k -algebra. Then the following are equivalent.*

- (a) *R is Cohen-Macaulay*
- (b) *There exists a homogeneous polynomial subring $k[x_1, \dots, x_r]$ such that R is finitely generated free module over $k[x_1, \dots, x_r]$.*
- (c) *If $k[x_1, \dots, x_r]$ is a homogeneous polynomial subring of R over which R is a finitely generated module then R is a free module over it.*

We use these equivalences as a main tool to prove Cohen-Macaulayness of a cohomology ring.

4.1. Cohomology ring of $(SL_2)_r$. Assume only in this part that $G = SL_2$. We prove that the cohomology ring $H^\bullet(G_r, k)$ is Cohen-Macaulay for each $r \geq 1$. This significantly improves a result of the author in [Ngo1] where he showed that the commutative ring $H^{2\bullet}(G_r, k)_{\text{red}}$ is Cohen-Macaulay. We first need a lemma.

Lemma 4.1.1. *Let S be a k -algebra on which B acts as algebra automorphisms¹. Suppose that U acts trivially on S , under the action of B , and S is regular as a commutative ring. Then the ring $\text{ind}_B^G S$ is Cohen-Macaulay.*

Proof. We have

$$\begin{aligned} \text{ind}_B^G S &= (k[G] \otimes S)^B \\ &\cong [(k[G] \otimes S)^U]^{B/U} \\ &\cong (k[G]^U \otimes S)^T \end{aligned}$$

Now it is not hard to compute that $k[G]^U$ is in fact a polynomial ring over 2 variables, see [Po, 2.1], so that it is a regular ring. As tensoring preserves regularity, we get $k[G]^U \otimes S$ is regular. Now since T is linearly reductive, the main result of Hochster-Robert in [HR] implies that the invariant ring $[k[G]^U \otimes S]^T$ is Cohen-Macaulay; hence completing our proof. \square

Remark 4.1.2. The Lemma 4.1.1 would not hold if the ring S was just Cohen-Macaulay. In fact, Hochster gave an explicit example in [Ho, p. 900] for a more general fact, that is, the invariant subring of a Cohen-Macaulay ring under a torus action is not Cohen-Macaulay. As a consequence, it is not true in general that the ring $\text{ind}_B^G R$ is Cohen-Macaulay provided that R is so. In other words, this provides a counter example for Conjecture 7.2.1 in an earlier paper of the author [Ngo1]. It remains interesting to know under what conditions the conjecture holds.

¹We usually call such S a B -algebra

Now we can tackle the Cohen-Macaulayness of the G_r -cohomology ring.

Theorem 4.1.3. *For each $r \geq 1$, the ring $H^\bullet(G_r, k)$ is Cohen-Macaulay, when $G = SL_2$.*

Proof. First recall from [Ngo1, Corollary 4.1.2] that the cohomology ring $H^\bullet(U_r, k)$ can be considered as a free module over the polynomial ring $k[x_1, \dots, x_r]$ where each x_i is of degree 2. Then a slight modification for Theorem 6.1.2 in [Ngo1] would give us that $H^\bullet(B_r, k)$ is a free module over $R = k[x_1^{p^{r-1}}, x_2^{p^{r-2}}, \dots, x_r]$. In other words, there is an isomorphism of R -modules as well as B -modules as follows

$$H^\bullet(B_r, k) \cong \bigoplus_{v \in \mathfrak{B}} v \otimes R$$

where \mathfrak{B} is the set of independent generators of $H^\bullet(B_r, k)$ as an R -module.

Following the strategy in [Ngo1, Theorem 4.3.1], one obtains the following isomorphism of algebras as well as G -modules

$$H^\bullet(G_r, k)^{(-r)} \cong \bigoplus_{v \in \mathfrak{B}} \text{ind}_B^G(v \otimes R)^{(-r)}.$$

Now it is easy to see that each graded k -algebra $\text{ind}_B^G(v \otimes R)^{(-r)}$ is the graded k -algebra $\text{ind}_B^G(R^{(-r)})$ shifted by $\deg(v)$. Hence it suffices to prove that the latter is Cohen-Macaulay.

As U trivially acts on \mathfrak{u}^* , it does the same on $H^\bullet(U_r, k)$. Since $R \subset H^\bullet(U_r, k)$, U acts trivially on R and so on $R^{(-r)}$. Now applying the lemma above, we get $\text{ind}_B^G(R^{(-r)})$ is Cohen-Macaulay. Thus the theorem follows from the fact that it is the direct sum of copies of $\text{ind}_B^G(R^{(-r)})$. \square

The Poincaré series associated to the cohomology ring $H^\bullet(G_r, k)$ is denoted by

$$p_{G_r}(t) = \sum_{i \geq 0} \dim H^i(G_r, k) t^i.$$

Next as a consequence of Theorem 4.1.3, one immediately has

Corollary 4.1.4. *Suppose $G = SL_2$. For each $r \geq 1$, the Poincaré series $p_{G_r}(t)$ associated to $H^\bullet(G_r, k)$ satisfies the Poincaré duality, i.e.*

$$p_{G_r}(1/t) = (-t)^d p_{G_r}(t).$$

Proof. Follows immediately from [ES, Theorem 12]. \square

4.2. Cohen-Macaulayness of U_r and B_r -cohomology. We are back to the assumption that G is a classical simple algebraic group. We show here that the Cohen-Macaulayness of $H^\bullet(U_r, k)$ implies the same for $H^\bullet(B_r, k)$.

Theorem 4.2.1. *Let G be a connected, reductive group and $r \geq 1$. If the algebra $H^\bullet(U_r, k)$ is a Cohen-Macaulay ring, then so is $H^\bullet(B_r, k)$.*

Proof. By Proposition 4.0.3, suppose $H^\bullet(U_r, k)$ is a free module over a polynomial ring \mathfrak{A} . In particular, let \mathcal{B} be a basis of $H^\bullet(U_r, k)$ over R , we then have

$$H^\bullet(U_r, k) = \bigoplus_{b \in \mathcal{B}} b \cup \mathfrak{A}.$$

Now for each $r \geq 1$, we have

$$\begin{aligned} H^\bullet(B_r, k) &\cong H^\bullet(U_r, k)^{T_r} \\ &\cong \bigoplus_{b \in \mathcal{B}} (b \cup \mathfrak{A})^{T_r} \end{aligned}$$

Each direct summand in the last item can be considered as a graded algebra \mathfrak{A}^{T_r} shifted by $\deg(b)$. On the other hand, \mathfrak{A}^{T_r} is Cohen-Macaulay by the main theorem of Hochster-Roberts in [HR]. So $H^\bullet(B_r, k)$ is a direct sum of Cohen-Macaulay rings; hence it is so. \square

4.3. Now we apply the properties of commuting varieties to show that both cohomology rings of B_r and G_r are not Cohen-Macaulay in general. In particular, we have the following

Proposition 4.3.1. *Suppose $p \geq h$. Then the rings $H^\bullet(U_r, k)$ and $H^\bullet(B_r, k)$ are not Cohen-Macaulay for the values of r in Proposition 3.2.2. Moreover, the cohomology ring $H^\bullet(G_r, k)$ is not Cohen-Macaulay for the values of r in Corollary 3.2.5*

Proof. We suppose on the contrary that both rings $H^\bullet(U_r, k)$ and $H^\bullet(B_r, k)$ are Cohen-Macaulay. Then they would be equidimensional (see for example [E, 18.10]). Hence, the famous result of Suslin-Friedlander-Bendel ([SFB1]) indicates that the variety

$$\text{MaxSpec } H^\bullet(U_r, k) = \text{MaxSpec } H^\bullet(B_r, k) = C_r(\mathbf{u})$$

would be equidimensional. This is impossible for values of r in Proposition 3.2.2. Similarly, we obtain the result for $H^\bullet(G_r, k)$. \square

Remark 4.3.2. This result provides many counter-examples for the Conjecture 7.2.2 in [Ngo1]. Even in the case $r = 2$, the cohomology rings $H^\bullet(U_2, k)$ and $H^\bullet(B_2, k)$ are also rare to be Cohen-Macaulay for the same reason. Indeed, Goodwin and Röhrle showed in their recent preprint [GR] that $C_2(\mathbf{u})$ is not equidimensional when \mathbf{u} has infinitely many B -orbits. Although in [GR] it is assumed that k has characteristic zero, the result is expected to be true for k of characteristics p when p is large enough.

5. COMPLEXITY OF FROBENIUS KERNELS

Keep the assumptions and notations as in the last section, we continue applying our results in the theory of commuting varieties to study cohomology for modules over Frobenius kernels.

5.1. Support varieties. Fix $r \geq 1$, suppose M is a G_r -module. We consider $\text{Ext}_{G_r}^\bullet(M, M)$ as a $H^{2\bullet}(G_r, k)$ -module via Yoneda product, set $J(M)$ the annihilator ideal in $H^{2\bullet}(G_r, k)$ for this action. Then the support variety for M , denoted by $V_{G_r}(M)$, is defined as the maximal spectrum of the quotient ring $H^{2\bullet}(G_r, k)/J(M)$. For further details and properties of support varieties, the reader can refer to [NPV, §2]. It was shown in [SFB2] that $V_{G_r}(M)$ is a closed conical subvariety of $C_r(\mathcal{N}_1)$. Similarly, if N is a B_r -module then $V_{B_r}(N)$ is a closed conical subvariety of $C_r(\mathbf{u}_1)$.

For each G_r -module M , the dimension of the support variety $V_{G_r}(M)$ is called the complexity of M . This notion can be interpreted as the growth rate of the series $\{\dim \text{Ext}_{G_r}^i(M, M)\}_{i=0}^\infty$, see [NPV, Theorem 2.2.2]. In particular, $c_{G_r}(k)$ is the Krull dimension of $H^\bullet(G_r, k)$.

5.2. We begin with some lower bound for the Krull dimension of the cohomology rings of B_r and G_r . This is an immediate consequence of Corollaries 3.2.3, 3.2.7, and Proposition 3.3.4.

Theorem 5.2.1. *For each $r \geq 1$, we have*

$$\dim H^\bullet(U_r, k) = \dim H^\bullet(B_r, k) \geq \begin{cases} r \lfloor \frac{n^2}{4} \rfloor & \text{if } \mathfrak{g} = \mathfrak{sl}_n, \\ r \frac{\ell^2 + \ell}{2} & \text{if } \mathfrak{g} = \mathfrak{sp}_{2\ell}, \\ r \frac{\ell^2 - \ell}{2} & \text{else,} \end{cases}$$

and

$$c_{G_r}(k) = \dim H^\bullet(G_r, k) \geq \begin{cases} (r+1) \lfloor \frac{n^2}{4} \rfloor & \text{if } \mathfrak{g} = \mathfrak{sl}_n, \\ (r+1) \frac{\ell^2 + \ell}{2} & \text{if } \mathfrak{g} = \mathfrak{sp}_{2\ell}, \\ (r+1) \frac{\ell^2 - \ell}{2} & \text{else.} \end{cases}$$

The equalities occur when G is of type A and $p = 2$.

5.3. Complexity for modules over Frobenius kernels. We investigate in this part some properties of the complexity for the simple module $L(\lambda)$ over G_r . First we need to set up some more notation in representation theory of G . Suppose Φ is the root system of G and Π is the set of simple roots in Φ . Let X be the weight lattice of G . Define

$$X^+ = \{\lambda \in X : (\lambda, \alpha^\vee) \geq 0, \text{ for all } \alpha \in \Pi\},$$

and

$$X_1 = \{\lambda \in X^+ : 0 \leq (\lambda, \alpha^\vee) < p, \text{ for all } \alpha \in \Phi^+\}.$$

We call them the set of dominant weights in X , and the set of p -restricted dominant weights in X^+ . All simple modules of G are $L(\lambda)$ with the highest weight $\lambda \in X^+$. Let $c = (\frac{\ell+1}{2})^2$ (resp. $\frac{\ell(\ell+1)}{2}$, $\frac{\ell^2}{2}$ or $\frac{\ell(\ell-1)}{2}$) if G is of type A_ℓ (resp. B_ℓ, C_ℓ or D_ℓ). Now combining results of Sobaje and those in commuting varieties, we have the following

Corollary 5.3.1. *Let $\lambda \in X_1$. Suppose $p > hc$. Then one has for each $r \geq 1$*

$$c_{G_r}(L(\lambda)) \geq \begin{cases} r \lfloor \frac{n^2}{4} \rfloor & \text{if } \mathfrak{g} = \mathfrak{sl}_n, \\ r \frac{\ell^2 + \ell}{2} & \text{if } \mathfrak{g} = \mathfrak{sp}_{2\ell}, \\ r \frac{\ell^2 - \ell}{2} & \text{else.} \end{cases}$$

Proof. Proposition 3.1 in [So] gives us

$$V_{G_r}(L(\lambda)) \supseteq V_{G_{r-1}}(k)$$

so that $c_{G_r}(L(\lambda)) \geq c_{G_{r-1}}(k)$. The last inequality and Theorem 5.2.1 prove our result. \square

Regarding upper bounds for $c_{G_r}(k)$, it is easy to see that there is always the following

$$c_{G_r}(k) = \dim C_r(\mathcal{N}_1) \leq \dim \mathcal{N}_1^r = r \dim \mathcal{N}_1 = r c_{G_1}(k).$$

However, this upper bound is not useful at all when $r > 1$. So finding sharper bounds for this amount could be an interesting problem. Given a finite dimensional G_r -module M , what we prove as follows is an upper bound for $c_{G_r}(M)$ in terms of $c_{B_r}(M)$.

Proposition 5.3.2. *For each $r \geq 1$, $c_{G_r}(M) \leq c_{B_r}(M) + \dim \mathfrak{u}$.*

Proof. It suffices to prove the inequality in terms of support varieties, that is

$$\dim V_{G_r}(M) \leq \dim V_{B_r}(M) + \dim \mathfrak{u}.$$

As the support variety $V_{B_r}(M)$ is a B -variety and $V_{G_r}(M) = G \cdot V_{B_r}(M)$, we then have the moment morphism $G \times^B V_{B_r}(M) \rightarrow V_{G_r}(M)$ is surjective. It follows that

$$\dim V_{G_r}(M) \leq \dim (G \times^B V_{B_r}(M)) = \dim(G/B) + \dim V_{B_r}(M) = \dim \mathfrak{u} + \dim V_{B_r}(M).$$

This proves our proposition. \square

Remark 5.3.3. When $M = k$, this upper bound is sharp and the equality occurs in the case of $r = 1$ and $p \geq h$. As a consequence, an upper bound of $c_{B_r}(M)$ gives that of $c_{G_r}(M)$.

Finally, our explicit study for restricted nilpotent commuting varieties over rank 2 Lie algebras provides some further information about the complexity for modules over Frobenius kernels in these small rank cases. The result below follows immediately from Section 3.4.

Corollary 5.3.4. *For $r \geq 1$ and M (or N) a finite dimensional G_r -module (or B_r -module), one has*

$$c_{B_r}(N) \leq \begin{cases} 2r + 1 & \text{if } G \text{ of } A_2, p > 2, \\ 2r & \text{if } G \text{ of } A_2, p = 2, \\ 3r & \text{if } G \text{ of } C_2, p = 3, \\ \max\{2r + 2, 3r\} & \text{if } G \text{ of } C_2, p > 3, \end{cases}$$

and

$$c_{G_r}(M) \leq \begin{cases} 2r + 4 & \text{if } G \text{ of } A_2, \\ \max\{2r + 6, 3r + 3\} & \text{if } G \text{ of } B_2 \text{ or } C_2. \end{cases}$$

In particular, the equality occurs when $M = k$.

Remark 5.3.5. This result generalizes the one of Kaneda et. al. in [KSTY] where they established the bounds for the Krull dimension of B_r -cohomology ring in the case $p > 2$,

$$2r \leq \dim H^*(B_r, k) \leq 2r + 1.$$

Our result is much more powerful as it not only gives an exact formula but also include the case $p = 2$.

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